Investigating functions: Domains and ranges

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I always find it of great use to challenge my more capable students with things that they have tended to take for granted. In this article, I would like to share a few of my favourites: questions that I have collected and used many times over the years, and which always result in valuable discussion, frequent bursts of insight, and more than the odd argument.

The role played by technology in the following questions is an interesting one. While most do not really depend upon technological aids, their use in most cases is beneficial. Students are provided with powerful investigative tools, which may include graph plotters, tables of values, even computer algebra systems. At times, however, lessons of caution may also be learned regarding when it may be best not to use technology! Certainly, in all cases, I would recommend asking students what they expect before they push the buttons: what do you expect the graph to look like? What do you expect to happen next? This is intelligent and strategic use of the tools, neither passive (waiting to be told or shown what to do) nor 'reflexive' (when students almost randomly apply operations and push buttons to see what will happen!).

One area in particular which students often tend to overlook in their senior years concerns the important understandings associated with domains and ranges of functions. I do suspect that many teachers brush over these concepts, assuming that their capable students already understand what is needed. In reality, this critical aspect of understanding functions and their properties can always do with a bit of a 'brush up'. Domains and ranges become critically important, of course, within the context of the study of inverse functions, and the following questions are probably best left until after the normal coverage is complete, when your students think they have it all under control.

Inside out functions

1. How many functions can you find which are their own inverses? How does a function affect its own inverse? Discuss the domains and ranges

of $\sin(\arcsin(x))$ and $\arcsin(\sin(x))$.

Begin, perhaps, by asking students to consider (preferably in pairs or small groups) what they know about inverse functions: what do they do? What may they look like? How do they behave? How do they act upon numbers? How do they act upon other functions? Inverse functions *undo* what the original function *does*: if $f(x)$ takes 2 and changes it to 5, then the inverse function will take 5 and change it into 2. You may be surprised at how few of your best students can articulate this essential understanding.

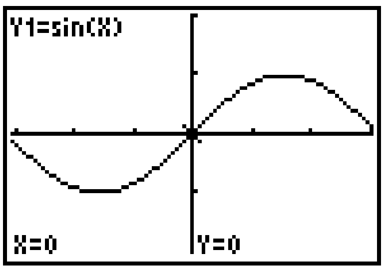
Graphically, the relationship of inverse functions with the identity function, $y = x$, is also critical. Some students will readily see that $y = x$ is, in fact, a function which is its own inverse — for others this will not be obvious. Are there, in fact, any others? If such functions exist, then what would they look like? In fact, they should be their own reflections in the line $y = x$. Investigate. (In fact, it seems unlikely that any other such functions do exist, since such a reflection for a continuous function would no longer be a function — also well worth probing with your students!)

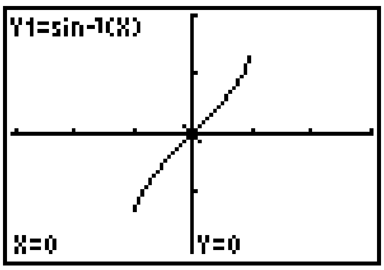
Algebraically, a function affects its own inverse by reducing it to x.

e.g. Consider $f(x) = 2x - 1$ and $g(x) = (x + 1)/2$.
Then $f(g(x)) = g(f(x)) = x$. (Check this!)

Again, this should not be new for your students, but do you notice an important assumption here? These are functions whose domains and ranges are unrestricted! Think carefully about cases not so well behaved: what are the implications?

Finally, turn your attention to the question in hand: what is the difference (if any) between $\sin(\arcsin(x))$ and $\arcsin(\sin(x))$? Students who immediately press the graph key on their graphics calculator are denied much of the fun of the chase here, and certainly are not encouraged to anticipate the result intellectually. So begin without any technological aids.





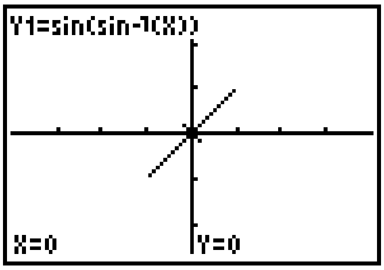
Perhaps consider the graph of $y = \sin(x)$ in terms of its domain (What numbers can go into this function? What does Sine like to eat?) and range (What sort of numbers come out of Sine?).

Now talk about domains and ranges of inverse functions: the range of the first function becomes the domain of the second (Always?) while the domain of the first may or may not help us with the range of the second!

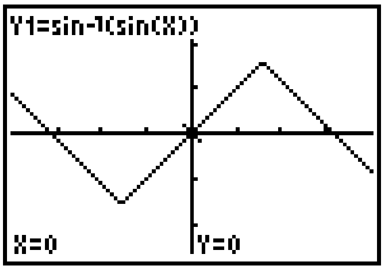
If students are struggling with the domain and range of the arcsine function, then you may like to show them the graph using a ViewScreen, and discuss what they observe.

So, now, we put them together: begin with the function $y = \sin(\arcsin(x))$. What sort of numbers can go into this function: look at the x. The domain of this composite function is that of the first function encountered: in this case, $\arcsin(x)$. That is, the domain of $\sin(\arcsin(x))$ is the range of $\sin(x)$. So the function only exists for values of x between -1 and 1.

Next, what values does $\arcsin(x)$ spit out (values between $-\pi/2$ and $\pi/2$), and what values does sine accept (anything!). No problems there. Finally, what values does sine spit out: values again between -1 and 1.



So $y = \sin(\arcsin(x))$ exists entirely in the square between -1 and 1 on both axes. Since the action of a function on its inverse should be to produce x, it is no surprise to see that it is that part of $y = x$ between -1 and 1.



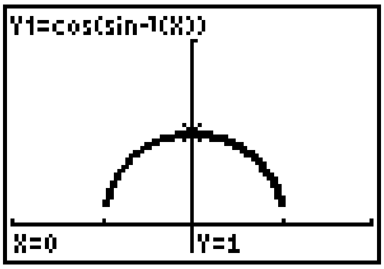
The surprise occurs when the function $y = \arcsin(\sin(x))$ is studied. What values can feed in (anything!). What comes out of Sine (values between -1 and 1, which are fine for arcsine). What values come out at the end: those between $-\pi/2$ and $\pi/2$. So the graph of this function appears to exist in a strip the full length of the x-axis, while trying to look like $y = x$.

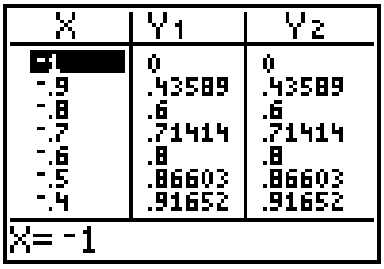
Few would have predicted this graph, and yet all should agree that it makes a certain kind of sense in terms of domains and ranges — if they understand domains and ranges, anyway!

Round and round we go

Let us continue our study with this question:

2. Consider the composite function
   $F(x) = \cos(\arcsin(x))$
   Is it really a semi-circle, or does it just look like one?
   Prove it!





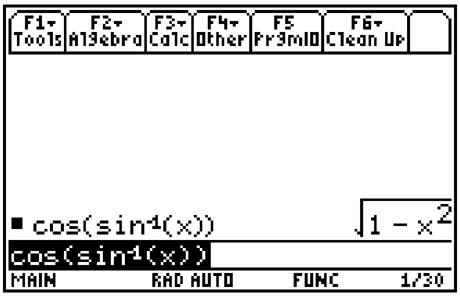
X	Y1	Y2
-1	0	0
-.9	.43589	.43589
-.8	.6	.6
-.7	.71414	.71414
-.6	.8	.8
-.5	.86603	.86603
-.4	.91652	.91652

X= -1

This one leads into consideration of the nature of proof in mathematics (if you are brave enough!). Here students are welcome to use the technology: look at the graph; compare it with the graph of the unit semi-circle; study the tables of values for both functions.

There is little question that the values are identical, but this is not enough for a proof. Perhaps a consideration of domains and ranges may be appropriate? At very least, it helps students to understand (or at least to feel that they understand better) the nature of this strange function.

An algebraic proof may or may not be beyond your students — certainly not if they have studied the fundamental trigonometric identity $\cos^2(x) + \sin^2(x) = 1$ from which we may readily derive the relationship $\cos(x) = \sqrt{(1 - \sin^2(x))}$. The rest follows if we accept that $\sin(\arcsin(x)) = x$ (which it certainly does between -1 and 1, where this function exists!). The value of this question, of course, lies in the search, not in the destination, and the algebraic result, if encountered too soon, detracts from a very worthwhile line of enquiry.

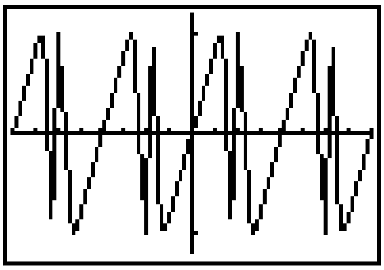


Interestingly, a CAS calculator, such as the TI-89 or *Voyage200*, is even a bigger party-pooper in such a question. Enter the function in question and it automatically simplifies it algebraically — directly to the result shown.

While this provides further powerful and convincing evidence, it is important that students come to appreciate the requirements of rigour which constitute a mathematical proof, and why they are necessary. This should be essential understanding for our more capable students and, rather than diminishing this, the availability of powerful technological tools can only help to make this difference clearer (while at the same time providing the tools needed to engage actively and with purpose in investigating such questions).

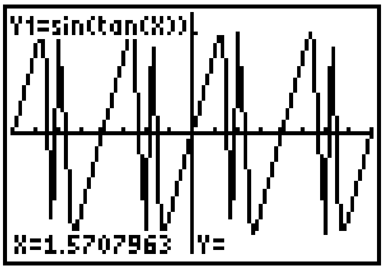
Finally, we might consider one more from our collection of domain and range questions:

Check the fuzzy bits



3. Study the graph of the composite function $\sin(\tan(x))$.
   What is happening at each of the fuzzy bits? If the function is periodic, then why do each of them appear different?

I do like this one, because it really needs the technological tools to properly investigate what is happening, and yet what is happening is a direct result of the nature of these tools themselves! Certainly, use domains and ranges, zoom in and zoom out, play with tables of values: the closer students look at this one, the more intractable it appears to become. It is no big deal that the *fuzzy bits* occur at those multiples of $\pi/2$ where the tangent function is undefined — that is hardly surprising. What is surprising is that the function does not appear to be undefined itself at those points — at least not in the sense that we are expecting to see (asymptotes soaring off to infinity?)



In fact, the function is undefined at these points, and this can be readily demonstrated using the trace mode on a graphic calculator. Using the TI-83 series, in TRACE mode, it is possible to go directly to any real value of x by simply typing that value directly: in this case, I entered $\pi/2$.

What makes this question interesting for me is the realisation that computers and calculators are, in fact, *rational number machines*: they cannot truly handle irrationals! When they plot (or trace) a graph, they do so by taking successive values of x, using very small (but rational) steps. Because of this, the calculator never really hits the value of $\pi/2$, or $-\pi/2$ or any of the other undefined points, and so they do not appear on our graph — or in our table of values! If the function appears not to be periodic, then the likely fault lies with the technology, not with the mathematics!

One further lesson of value for our students: do not always believe what the calculator or computer tells you! Always weigh it up against your own understanding and expectations, and always seek to justify such a result mathematically!

Look at the BIG Picture

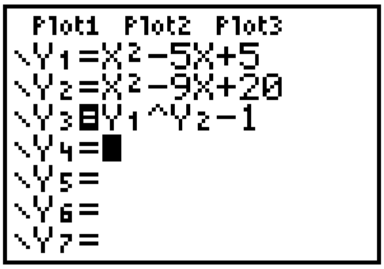
Finally, a classic problem which appears much harder than it is.

4. Solve: $\left(x^2-5x+5\right)^{x^2-9x+20}=1$

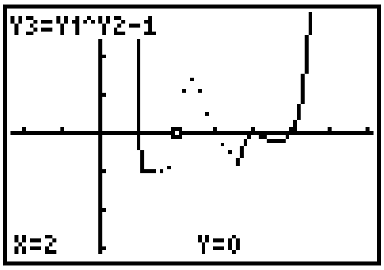
   How many solutions did you find? Are you sure you have them all?

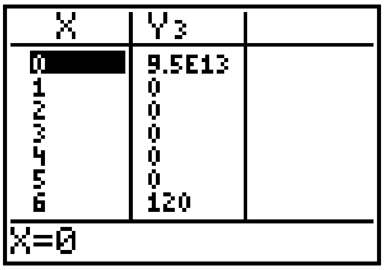
One of the differences between those students who are successful at mathematics and the rest lies in the ability to not be distracted by the little things — to stand back from a question and look at what it is actually asking. In this case, something to the power of something equals one. Once students view the question in this way, it tends to fall open. It usually does not take a capable student (working with other capable students) too long to find solutions of 1, 4 and 5.

Then the real fun begins, because the real question here is not about finding some answers: it is about ascertaining that we have all the answers!



Turning them loose upon the technology invariably leads to graphing (and interesting questions about how best to graph an equation!). There is value, I believe, in encouraging students to break the problem down into its component parts, even in setting up the function (as shown).





X	Y_3
0	9.5E13
1	0
2	0
3	0
4	0
5	0
6	120

X=0

Tracing the graph reveals some interesting features, including undefined values — and more solutions!

In fact, it is a sobering experience for those students who automatically reach for the graph every time to find that, in this case, the table of values reveals all five solutions immediately! But justifying these solutions is a different matter. Where do the 2 and the 3 come from? Once again, this is a source for discussion, group work and some frustration, until someone realises that there is another way to make 1 from an exponent, and this involves an even power of -1!

If, however, we were so easily deceived in these simple matters, then some caution should be observed before assuming that, *obviously*, we now have *all* the solutions. To satisfy this final part of the question, we should refer back to the graph, which strongly suggests that the function increases to infinity outside the range of solutions observed. The most convincing argument, however, is that derived once more from considerations of domain and range.

If we think about the domains and ranges of the two quadratics that make up this function, then there can be little doubt that as x approaches infinity in both positive and negative directions, that the function in questions should always approach positive infinity, and so we have good reason to believe that all solutions occur between the values of 1 and 5.

Interestingly this question is not algebraically difficult — it just appears that way, and students will need to be encouraged to ‘have a go’. Their first impulse is invariably to reject the function as ‘too difficult’. Group work and access to technology serve useful scaffolding functions in this case, as does the encouragement and positive reinforcement of the teacher!

For other questions which may serve to challenge and to infuriate your students, you are welcome to visit

http://smarnold.freewebpage.org/functions.htm

and choose ‘Some Things to Think About: Investigating Functions and Graphs’.